Differentiation of Integrals: Frontiers and Prospectives

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Fundamental Theorem of Calculus

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

Let $F(x) = \int_0^x f(t)\,dt$

Then $F'(x) = f(x)$.

Proof

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

$$= f(x). \qquad \blacksquare$$

the FTC is about limits of averages of f !

Lebesgue Differentiation Theorem

Let $f \in L^1(\mathbb{R}^n)$. Then for a.e. $x \in \mathbb{R}^n$,

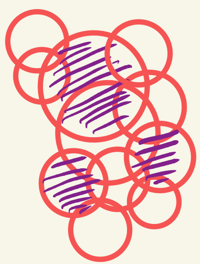
$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f = f(x).$$

Vitali Covering Lemma

Let $\{B_j\}_{j=1}^{N}$ be a finite collection of open balls in $\mathbb{R}^n$. Then there exists a subcollection $\{\tilde{B}_j\} \subset \{B_j\}$ satisfying

i) the $\tilde{B}_j$ are pairwise disjoint

ii) $|\cup \tilde{B}_j| \geq 3^{-n} |\cup B_j|$.



Def. (Centered Hardy-Littlewood Maximal Operator)

$$M_{HL}^{c} f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f|$$

Theorem

$$\left|\{x \in \mathbb{R}^n : M_{HL}^{c} f(x) > \alpha\}\right| \leq \frac{3^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$$

$\leq \frac{c}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$? w/ $c \neq c(n)$

weak type (1,1) inequality

We can use this to prove the LDT.

Paradigm:

Covering Lemma $\Longrightarrow$ Weak Type Bound of Maximal Operator $\overset{\text{N. Wiener 1939}}{\Longrightarrow}$ Differentiation Theorem

$\Longleftrightarrow$

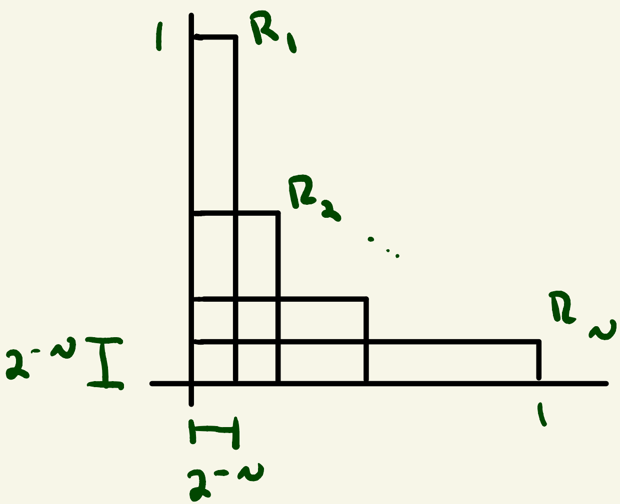
Córdoba - Fefferman
1975

$\Longleftarrow$

Stein - Nikishin
w/ translation invariance
1961

What about averages with respect to rectangles?

The Vitali Covering Lemma does not hold for rectangles!



Bohr Staircase

(Harold Bohr)

$$\frac{\left|\bigcup_{j=1}^{N} R_j\right|}{|R_i|} \sim N$$

Theorem (Nikodym; Busemann-Feller)

Let $\mathcal{F}$ be the family of <u>all</u> rectangular parallelepipeds in $\mathbb{R}^n$.

There exists $f \in L^\infty(\mathbb{R}^n)$ such that

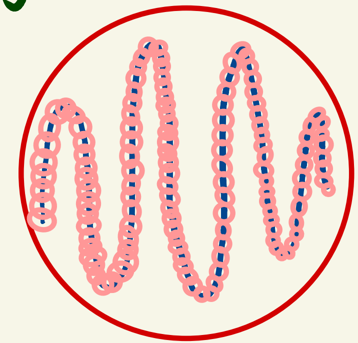
$$\lim_{\substack{x \in R \in \mathcal{F} \\ \operatorname{diam} R \to 0}} \frac{1}{|R|} \int_R f = f(x)$$

does not hold a.e.

A <u>Besicovitch</u> <u>set</u> in $\mathbb{R}^n$ is a set that contains a unit segment in every direction.

Recall: If $E \subset \mathbb{R}^n$, $\delta > 0$,

$$H_r^\delta(E) = \inf_{\substack{E \subset \cup B_j \\ \operatorname{diam} B_j < r}} \sum (\operatorname{diam} B_j)^\delta$$



$$H^\delta(E) = \lim_{r \to 0^+} H_r^\delta(E)$$

The Hausdorff dimension of E is

$$\inf\{\delta > 0 : H^\delta(E) < \infty\}$$

$$\boxed{\begin{array}{l}\text{Kakeya Conjecture:} \\ \quad \text{Let } S \text{ be a Besicovitch set in } \mathbb{R}^n. \\ \text{Then the Hausdorff dimension of } S \text{ is } n.\end{array}}$$

$n = 1, 2$ ✓ (Davies) 1971

$n \geq 3$ partial results due to Wolff, Bourgain, Tao, Katz, and others

e.g. $n > 12$ $\quad \frac{6(n-1)}{11} + 1$

Strong Differentiation

Fund. Math. 1935

Theorem (Jessen, Marcinkiewicz, Zygmund)

If $f \in L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$

(in fact if $\int_{\mathbb{R}^n} |f| \log(2 + |f|)^{n-1} < \infty$)

then

$$\lim_{\substack{x \in R \\ \operatorname{diam} R \to 0 \\ \text{sides } R \parallel \text{axes}}} \frac{1}{|R|} \int_R f = f(x) \text{ a.e.}$$

Pf $\sim$ Iteration of 1-D estimates.

Let $\mathcal{B}$ be a collection of open bounded sets that covers $\mathbb{R}^n$.

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f| .$$

Annals 1975

Thm. (A. Córdoba, R. Fefferman)

Let $1 < p < \infty$.

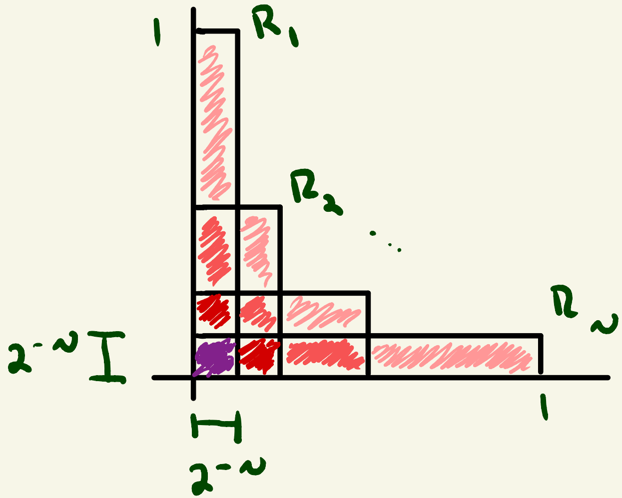
i.e. M_B is of weak type (p,p)

$$|\{x \in \mathbb{R}^n : M_B f(x) > \alpha\}| \leq C \left(\frac{\|f\|_p}{\alpha} \right)^p$$

iff there exist $0 < c_1 < c_2 < \infty$ so that given $\{R_j\}_{j=1}^{N} \subset \mathcal{B}$, there exists $\{\tilde{R}_j\} \subset \{R_j\}$ satisfying

i) $|\cup \tilde{R}_j| \geq c_1 |\cup R_j|$

ii) $\| \sum \chi_{\tilde{R}_j} \|_{L^{p'}} \leq c_2 |\cup R_j|^{1/p'}$.



$$|\blacksquare| = 2^{-2n} \lesssim \frac{1}{n \cdot 2^{n}} \left|\cup R_n\right|$$

Problem : Suppose M_B is of weak type $(1,1)$,

i.e.
$$\left|\{x \in \mathbb{R}^n : M_B f(x) > \alpha\}\right| \leq \gamma \frac{\|f\|_1}{\alpha}.$$

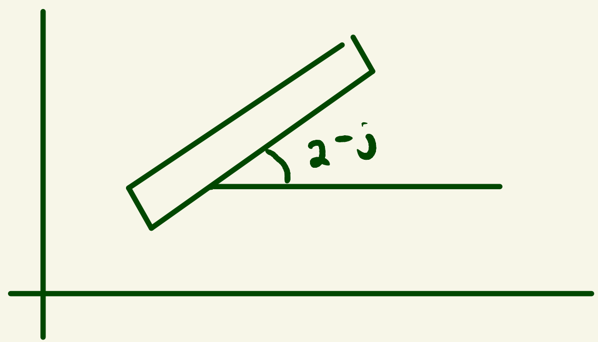
Does there exist $0 < c, C < \infty$ so that, given $\{R_j\}_{j=1}^N \subset \mathcal{B}$, there exists $\{\tilde{R}_j\} \subset \{R_j\}$ such that

i) $|\cup \tilde{R}_j| > c\,|\cup R_j|$

ii) $\|\sum \chi_{\tilde{R}_j}\|_\infty \leq C$?

lacunary

Let $\mathcal{R}_{lac}$ denote the set of rectangles in $\mathbb{R}^2$ oriented in a direction of the form 2^{-j}



Thm (A. Córdoba, R. Fefferman, Strömberg)

Let $$M_{lac} f(x) = \sup_{x \in R \in \mathcal{R}_{lac}} \frac{1}{|R|} \int_R |f| .$$

If $2 < p \le \infty$,

$$\| M_{lac} f \|_p \le C_p \|f\|_p .$$

Remark $1 < p \le 2$ ✓ Nagel-Stein-Wainger

(Fourier analytic techniques)

– no known proof via covering lemma for $1 < p \le 2$!

1 lac.

2 lac.

$\vdots$

$\{2^{-k}\}$

$\{2^{-k}+2^{-\ell}\}$

Ann. Inst. Fourier 1981

<u>Thm</u> (Sjögren, Sjölin)

Let Ω be an N-lacunary set in $[0,1]$ and let $\mathcal{B}_\Omega$ denote the set of rectangles in $\mathbb{R}^2$ oriented in a direction in Ω. The associated maximal operator $M_{\mathcal{B}_\Omega}$ satisfies

$$\|M_{\mathcal{B}_\Omega} f\|_p \le C_p \|f\|_p \qquad 1 < p \le \infty$$

AJM 2015

<u>Rk</u> Analogues for $n \geq 3$ by Parcet, Rogers

Thm. (Bateman Duke 2009)

Let Ω be a set of directions in $[0,1]$.

The associated directional maximal operator M_{B_Ω} is bounded on $L^p(\mathbb{R}^2)$ for $1 < p \leq \infty$ iff Ω can be covered by finitely many N-lacunary sets. Otherwise M_{B_Ω} is unbounded on $L^p(\mathbb{R}^2)$ for every $1 \leq p < \infty$.

What is $n \geq 3$ dimensional analogue?

Let $\mathcal{B}$ be a collection of open bounded sets in $\mathbb{R}^n$. $\mathcal{B}$ is called a **density basis** if for every measurable set $E \subset \mathbb{R}^n$ for a.e. x

$$\lim_{j\to\infty} \frac{1}{|R_j|} \int_{R_j} \chi_E = \chi_E(x)$$

holds whenever $\{R_j\}$ is a sequence of sets in $\mathcal{B}$ containing x whose diameters are tending to 0.

Let

$$C_{\mathcal{B}}(\alpha) = \sup_{0<|E|<\infty} \frac{1}{|E|} \left| \{ x : M_{\mathcal{B}} \chi_E(x) > \alpha \} \right|$$

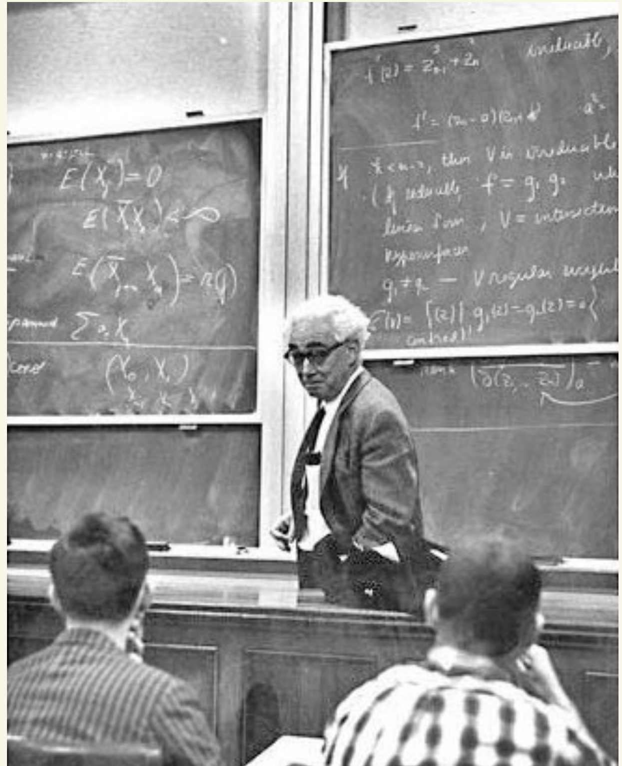
Thm (Busemann-Feller Fund. Math 1934)

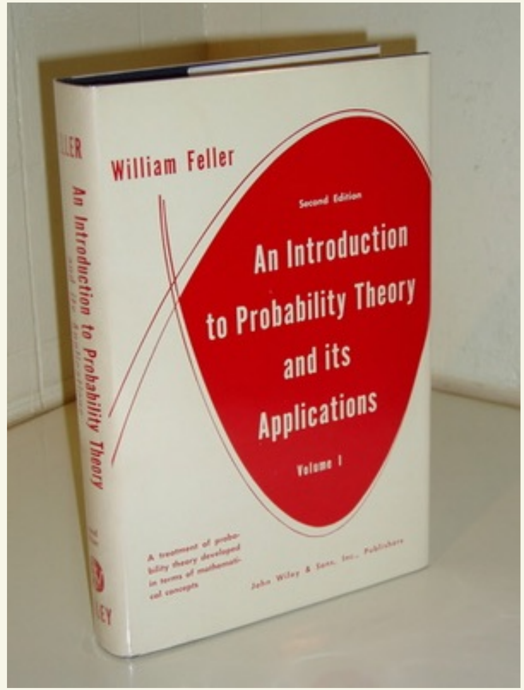
Let $\mathcal{B}$ be a homothecy invariant (invariant under translations and dilations) collection of open bounded sets that covers $\mathbb{R}^n$.

$\mathcal{B}$ is a density basis

iff

$C_{\mathcal{B}}(\alpha) < \infty$ for every $\alpha > 0$.

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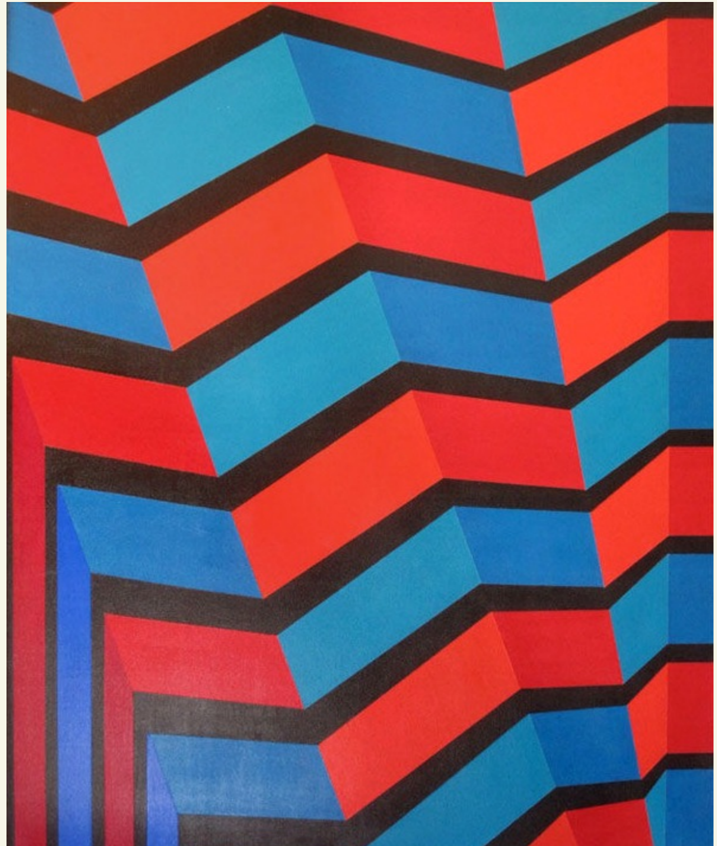


Halo Conjecture

Let $\mathcal{B}$ be a homothecy invariant density basis of sets in $\mathbb{R}^n$, and suppose

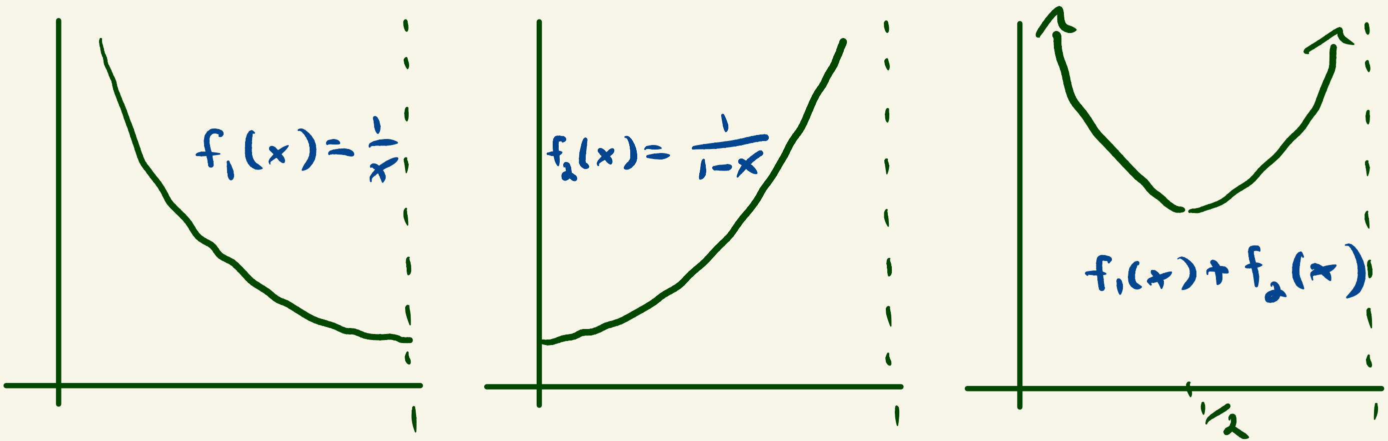
$$\left|\left\{x \in \mathbb{R}^n : M_{\mathcal{B}} \chi_E(x) > \alpha\right\}\right| \leq \int_{\mathbb{R}^n} \varphi\left(\frac{\chi_E}{\alpha}\right)$$

i.e. $C_{\mathcal{B}}(\alpha) \leq \varphi\left(\frac{1}{\alpha}\right)$

for some convex increasing $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) = t$ for $0 \leq t \leq 1$. Then

$$\left|\left\{x \in \mathbb{R}^n : M_{\mathcal{B}} f(x) > \alpha\right\}\right| \leq C \int_{\mathbb{R}^n} \varphi\left(\frac{|f|}{\alpha}\right).$$

The Halo Conjecture is hard, in part, because Weak L^1 is not a Banach space!
(No triangle inequality.)



$\|f_1\|_{WL^1} = 1$ $\qquad$ $\|f_2\|_{WL^1} = 1$ $\qquad$ $\|f_1 + f_2\|_{WL^1} = 4$

Theorem (M. de Guzmán)

If the Halo Conjecture is true, then for any homothecy invariant density basis $\mathcal{B}$ we have

$$\| M_{\mathcal{B}} f \|_p \leq C_p \| f \|_p$$

for sufficiently large p.

Thm (H., Stokolos TAMS 2009)

Let $\mathcal{B}$ be a homothecy invariant density basis consisting of convex sets in $\mathbb{R}^n$. Then

$$\| M_{\mathcal{B}} f \|_p \leq C_p \| f \|_p$$

for sufficiently large p.

Thm. (H., Stokolos 2022)

(") in $\mathbb{R}^2$. Then

$$\| M_{\mathcal{B}} f \|_p \leq C_p \| f \|_p$$

for all $1 < p \leq \infty$.

Open: $\mathbb{R}^n$ analogue for $n \geq 3$.

proof involves Bernoulli percolation, builds on work of Bateman, Katz, Laba, Lyons (sticky maps)

Suppose $\mathcal{B}$ is a translation invariant collection of sets in $\mathbb{R}^n$. When is $\mathcal{B}$ a density basis?

Notation: $\mathcal{B}_r = \{ R \in \mathcal{B} : \text{diam}(R) < r \}$

Thm. (H., Parissis Fund. Math 2018)

$\mathcal{B}$ is a density basis iff, given $\alpha > 0$, there exists $r = r(\alpha) > 0$ so that $C_{\mathcal{B}_r}(\alpha) < \infty$.

Problem: If $\mathcal{B}$ is a t.i. density basis, does there exist $r > 0$ so that $C_{\mathcal{B}_r}(\alpha) < \infty$ for all $0 < \alpha < 1$?

Application to PDE's

Let $f \in L^p(\mathbb{R}^n)$ and let $u(x,y)$ be its Poisson extension to

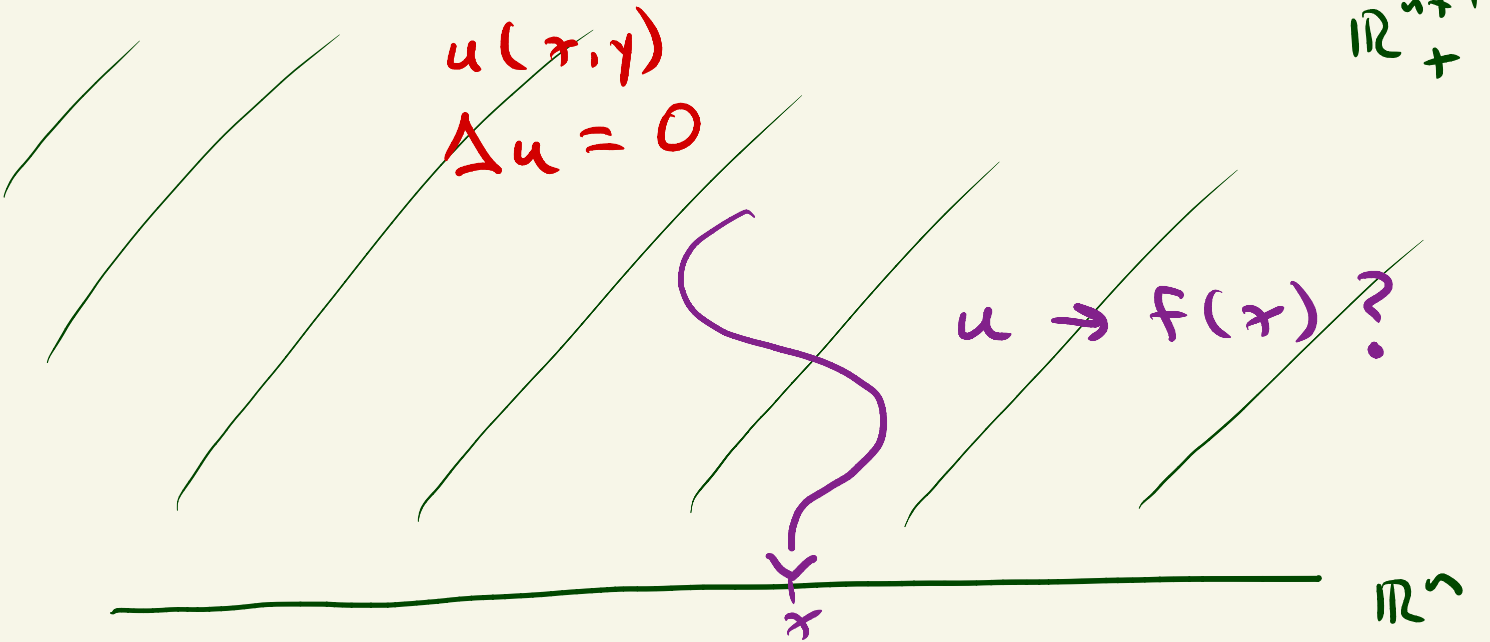
$$\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\ y > 0\}$$

given by

$$u(x,y) = \int_{\mathbb{R}^n} P_y(x) f(x-t)\,dt$$

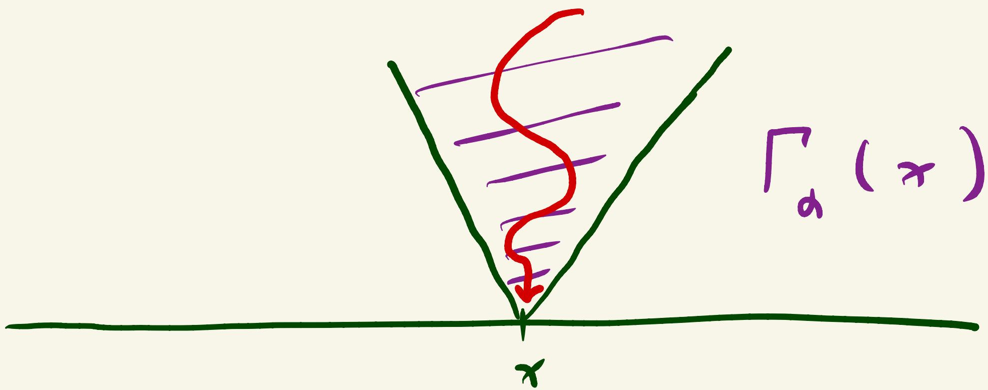
where $P_y(x)$ is the Poisson kernel

$$P_y(x) = \frac{c_n y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}.$$



boundary data $f(x) \in L^p(\mathbb{R}^n)$

$1 \le p \le \infty$



$$\Gamma_\alpha(x) = \{(s,t) \in \mathbb{R}^{n+1}_+ : |s-x| < \alpha t\}$$

Theorem (Fatou, 1906)

Let $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$

and let $u(x,y)$ be its Poisson integral.

Then

$$\lim_{\substack{(s,t) \to (x,0) \\ (s,t) \in \Gamma_\alpha(x)}} u(s,t) = f(x)$$

holds for a.e. x.

(Nontangential convergence)

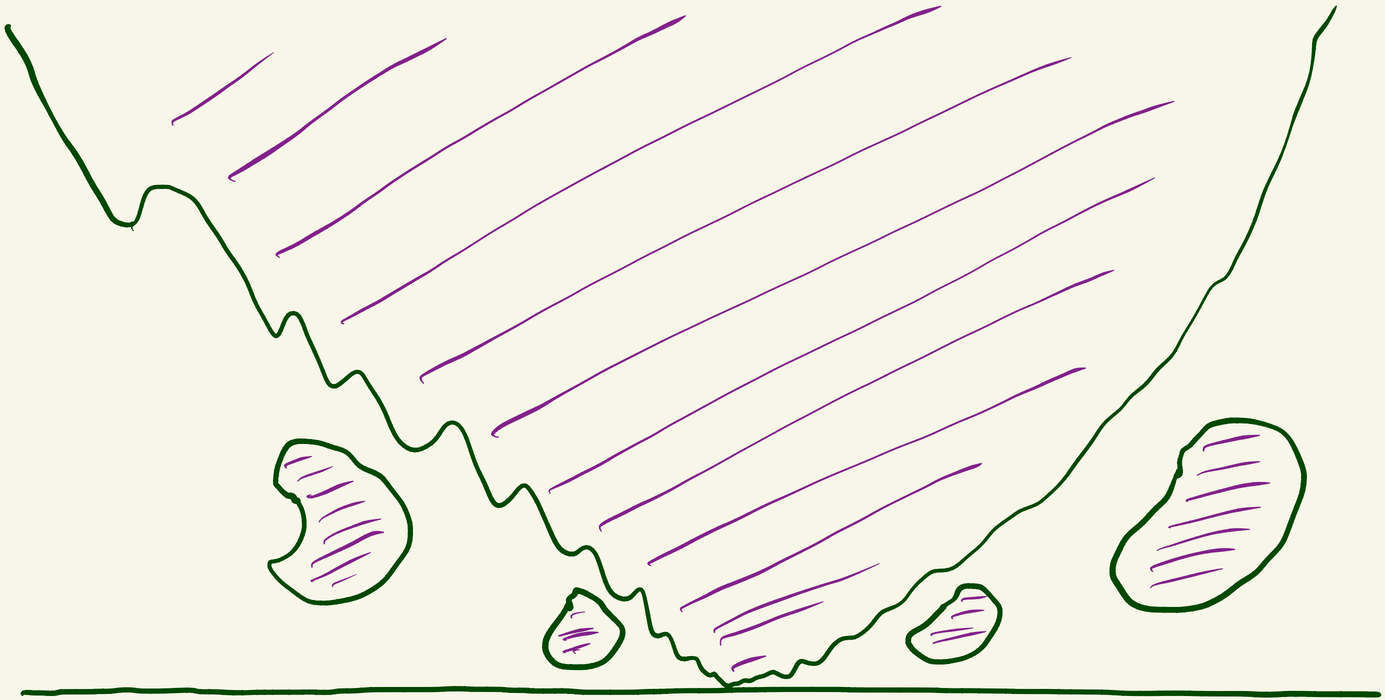
Thm (H., Parissis Studia Math 2020)

Let $\Omega \subset \mathbb{R}^{n+1}_+$ be such that $(0,0) \in \overline{\Omega}$. Suppose that for every $f \in L^\infty(\mathbb{R}^n)$, for a.e. $x \in \mathbb{R}^n$ we have

$$\lim_{\substack{(s,t)\in\Omega \\ (s,t)\to(0,0)}} u(x+s,t) = f(x)$$

holds. Then given $f \in L^p(\mathbb{R}^n)$ for some $1 \le p \le \infty$, for a.e. $x \in \mathbb{R}^n$ we have

$$\lim_{\substack{(s,t)\in\Omega \\ (s,t)\to(0,0)}} u(x+s,t) = f(x).$$



Solyanik Estimates

$$M_{HL} f(x) = \sup_{x \in B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f|$$

$$C_{HL}(\alpha) = \sup_{0 < |E| < \infty} \frac{1}{|E|} \left| \{ x \in \mathbb{R}^n : M_{HL} \chi_E(x) > \alpha \} \right|$$

$$\left(\leq \frac{3^n}{\alpha} \right)$$

Theorem (H., Parissis 2014)

$$C_{HL}(\alpha) - 1 \lesssim \left(\frac{1}{\alpha} - 1 \right)^{\frac{1}{n+1}}$$

for $\alpha \in (1-\varepsilon, 1)$

can this be improved to $\frac{1}{n}$ or $\frac{2}{n+1}$?

$$M_s f(x) = \sup_{\substack{x \in R \\ \text{sides } R \parallel \text{axes}}} \frac{1}{|R|} \int_R |f|$$

$$C_s(\alpha) = \sup_{0 < |E| < \infty} \frac{1}{|E|} \left| \left\{ x \in \mathbb{R}^n : M_s \chi_E(x) > \alpha \right\} \right|$$

<u>Theorem</u> (Solyanik 1993)

$$C_s(\alpha) - 1 \sim \left(\frac{1}{\alpha} - 1 \right)^{1/n} \leftarrow \text{sharp!}$$

Theorem (H., Parissis Adv. Math 2015)

$$C_{HL}(\alpha) \in C^{1/n}(0,1)$$

Pf: Iterated halo argument using Solyanik estimates.

$$C_S(\alpha) \in C^{1/n}(0,1)$$

Are C_{HL}, C_S in $C^\infty(0,1)$?

$$C_B(\alpha) \leq C_B\left(\alpha\left(1+\frac{\delta}{2^n}\right)\right) C_B(1-\delta) \qquad 0 < \delta < 1-\alpha$$

B convex basis in $\mathbb{R}^n$.

Let $\mathcal{B}$ be a homothecy invariant collection of convex sets in $\mathbb{R}^n$, and let

$$M_{\mathcal{B}} f(x) = \sup_{x \in R \in \mathcal{B}} \frac{1}{|R|} \int_R |f| .$$

Let $$C_{\mathcal{B}}(\alpha) = \sup_{0 < |E| < \infty} \frac{1}{|E|} \left| \{ x \in \mathbb{R}^n : M_{\mathcal{B}} \chi_E(x) > \alpha \} \right| .$$

Suppose $\mathcal{B}$ is a density basis (i.e. $C_{\mathcal{B}}(\alpha) < \infty$ for all $0 < \alpha < 1$.)

Problem: Is $\lim_{\alpha \to 1^-} C_{\mathcal{B}}(\alpha) = 1$?

– analogous problem for translation invariant bases.

Thank you for listening!